CONCERNING THE DISCONTINUOUS SOLUTION IN THE PROBLEM OF THE MINIMUM SURFACE OF REVOLUTION

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1. In the problem of minimizing the integral

$$J = \int_{t_0}^{t_1} y\sqrt{x'^2 + y'^2}\, dt,$$

where the admissible curves are all "ordinary"* curves which can be drawn in the upper half plane ($y \geqq 0$) from the given point A to the given point B, Euler's differential equation has two solutions:

(1) the catenaries

$$x = t \;; \quad y = m \operatorname{ch} \frac{t - x_0'}{m}; \tag{1}$$

(2) the straight lines

$$x = a \;; \quad y = t. \tag{2}$$

The latter solution leads to the well-known "discontinuous solution" of the problem, first noticed by Goldschmidt,† which consists (see Figure 3) of

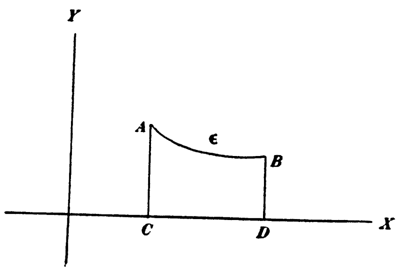

FIG. 3.

the perpendicular AC to the x-axis, the segment CD of the x-axis, and the perpendicular DB.

* In the terminology of Bolza, *Lectures on the Calculus of Variations*, p. 117.
† *Göttingen Prize Essay*, 1831.

The segment CD is not itself an extremal but it satisfies the boundary conditions for a minimum arising from the first variation (see Bolza, *Lectures on the Calculus of Variations*, p. 153).

Todhunter* has given a simple geometrical sufficiency proof that $I_{ACDB} < I_{AB}$ for every admissible curve $AB = \mathfrak{C}$ whose length is greater than $|AC| + |DB|$ (see Figure 3).

We now consider a catenary of the system (1) passing through the point A (x_0, y_0), namely:

$$y = m \operatorname{ch} \frac{x - x_0'}{m}, \tag{3}$$

where

$$y_0 = m \operatorname{ch} \frac{x_0 - x_0'}{m}; \tag{4}$$

and take on this catenary the point B (x_1, y_1).

We propose to compare the areas of the surfaces of revolution generated by the arc AB of the catenary on the one hand and by the discontinuous solution $ACDB$ on the other hand.

The value of the latter is

$$S_D = \pi(y_0^2 + y_1^2) = \pi\, m^2 \left\{ \operatorname{ch}^2 \frac{x_0 - x_0'}{m} + \operatorname{ch}^2 \frac{x_1 - x_0'}{m} \right\},$$

and the value of the former

$$S_C = \frac{\pi m}{2} \left\{ 2(x_1 - x_0) + m \operatorname{sh} \frac{2(x_1 - x_0')}{m} - m \operatorname{sh} \frac{2(x_0 - x_0')}{m} \right\}.$$

If we introduce the abbreviations

$$u_0 = \frac{x_0 - x_0'}{m}, \quad u_1 = \frac{x_1 - x_0'}{m} \tag{5}$$

we obtain for the difference

$$\begin{aligned} S_D - S_C &= \pi m^2 \left\{ (2u_0 + \operatorname{sh} 2u_0 + 2 \operatorname{ch}^2 u_0) - (2u_1 + \operatorname{sh} 2u_1 - 2 \operatorname{ch}^2 u_1) \right\} \\ &= \pi m^2 \left\{ (2u_0 + 1 + e^{2u_0}) - (2u_1 - 1 - e^{-2u_1}) \right\}. \end{aligned}$$

As the point B moves along the catenary in the direction of the increasing

* Todhunter, *Researches on the Calculus of Variations*, §§64, 65

x, the quantities m, x_0, x_0', and therefore also u_0, remain constant, while u_1 increases. We therefore consider the function

$$\phi\ (u_1) = 2u_1 - 1 - e^{-2u_1},$$

whence
$$\phi'\ (u_1) = 2(1 + e^{-2u_1}) > 0.$$

Then $\phi\ (u_1)$ increases with u_1 and therefore $S_D - S_C$ decreases with u_1. When B coincides with A, $u_1 = u_0$ and $S_D - S_C = 4\pi m^2 \operatorname{ch}^2 u_0 > 0$; as x_1 and therefore u_1 approaches $+\infty$, $S_D - S_C$ approaches $-\infty$. Hence there exists one and but one value of u_1 for which $S_D - S_C = 0$; that is:

There exists one and but one point A'' on the catenary for which the discontinuous solution has the same value as the catenary solution, i. e. for which

$$2u_0 + 1 + e^{2u_0} = 2u_1 - 1 - e^{-2u_1}. \tag{6}$$

If B lies between A and A'', the catenary solution has a smaller value than the discontinuous solution, while if B lies beyond A'', the discontinuous solution has the smaller value.

The point A'' always lies between A and its conjugate A'; for by Lindelöf's theorem* at the point A' (see Figure 4)

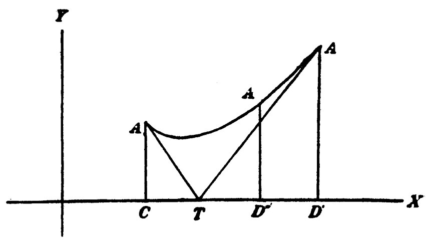

FIG. 4.

the area (AA')† is equal to the area generated by the two tangents AT and $A'T$ which moreover intersect on the x-axis. But the area $(ACD'A')$ is less than the area (ATA'), and therefore also less than the area (AA'). Hence according to the above result the point A' must lie beyond A''.

2. If we construct the point A'' for every catenary of the set (1) through the point A, the points A'' describe a curve which we propose to study in this section.

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* Hancock, "On the number of catenaries that may be drawn through two given points," ANNALS OF MATHEMATICS, ser. 1, vol 10, p. 159, §16; also *Calculus of Variations*, chapter III.

† I. e, the area generated by the arc AA'.

The coordinates of the point A'' for a given catenary are determined by the equations

$$x_1 = x_0 + y_0 \frac{u_1 - u_0}{\operatorname{ch} u_0}, \tag{7}$$

$$y_1 = y_0 \frac{\operatorname{ch} u_1}{\operatorname{ch} u_0}, \tag{8}$$

when u_0 and u_1 are connected by the relation

$$2u_0 + 1 + e^{2u_0} = 2u_1 - 1 - e^{-2u_1}. \tag{9}$$

Hence we obtain the required locus if we eliminate u_0 from the equations (7), (8), (9) or else express u_0, u_1 in terms of a variable t. It is convenient to choose for t the common value of the two sides of equation (9):

$$t = 2u_0 + 1 + e^{2u_0} = 2u_1 - 1 - e^{-2u_1}. \tag{10}$$

We shall now consider the two curves in the u-t plane represented by equation (10). In the first place,

$$\frac{dt}{du_0} = 2(1 + e^{2u_0}) > 0;$$

therefore t increases with u_0. Secondly, (11)

$$\frac{d^2t}{du_0^2} = 4e^{2u_0} > 0;$$

therefore the curve is convex to the u_0-axis. Again, (12)

$$\frac{dt}{du_1} = 2(1 + e^{-2u_1}) > 0;$$

therefore t increases with u_1. Finally, (13)

$$\frac{d^2t}{du_1^2} = -4e^{-2u_1} < 0;$$

therefore the curve is concave to the u_1-axis. (14)

Since both functions increase with u_0, u_1 respectively, it follows that u_0 and u_1 are single valued functions of t. In order to obtain corresponding values of u_0 and u_1 we construct the two graphs upon the same set of axes from the following table:

TABLE I

u_0	t	u_1	t
.00	2.00	.00	−2.00
.20	2.89	.20	−1.27
.40	4.03	.40	−0.65
.60	5.52	.60	−0.10
.80	7.55	.80	+0.40
1.00	10.39	1.00	+0.86
:	:	1.50	1.95
∞	∞	2.00	2.98
−.20	1.27	2.50	3.99
−.40	0.65	3.00	4.99
−.60	0.10	asymptotic	to line
−.80	−0.40	$t = 2u_1 - 1$	
−1.00	−0.86	for positive	values of u_1
−1.50	−1.95	−.20	−2.89
−2.00	−2.98	−.40	−4.03
asymptotic	to straight	−0.60	−5.52
line	$t = 2u_0 + 1$	−0.80	−7.55
for negative	values of u_0	−1.00	−10.39
		:	:
		$-\infty$	$-\infty$

The left branch is the curve $t = 2u_0 + 1 + e^{2u_0}$, while the right branch is the curve $t = 2u_1 - 1 - e^{-u_1}$. (See Figure 5.)

We next compute the derivatives dx_1/dt and dy_1/dt by means of equations (7), (8), (10), (12), and (14). First,

$$\frac{dx_1}{dt} = y_0 \frac{\operatorname{ch} u_0\left(\dfrac{du_1}{dt} - \dfrac{du_0}{dt}\right) - (u_1 - u_0) \operatorname{sh} u_0 \dfrac{du_0}{dt}}{\operatorname{ch}^2 u_0} \tag{15}$$

$$= y_0 \frac{\operatorname{ch}^2 u_0 \operatorname{sh} u_1(\operatorname{ch} u_1 + \operatorname{sh} u_1) - \operatorname{ch}^2 u_1 \operatorname{sh} u_0(\operatorname{ch} u_0 - \operatorname{sh} u_0)}{4 \operatorname{ch} u_1 \operatorname{ch}^3 u_0(\operatorname{ch} u_1 + \operatorname{sh} u_1)} \tag{16}$$

$$= y_0 \frac{\operatorname{ch}^2 u_0 \operatorname{sh}^2 u_1 + \operatorname{ch}^2 u_1 \operatorname{sh}^2 u_0 + \operatorname{ch} u_1 \operatorname{ch} u_0 \operatorname{sh}(u_1 - u_0)}{4 \operatorname{ch} u_1 \operatorname{ch}^3 u_0(\operatorname{ch} u_1 + \operatorname{sh} u_1)} \tag{17}$$

Here the hyperbolic cosine is always positive, y_0 is positive, and

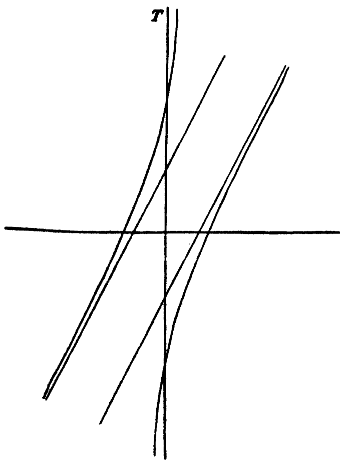

FIG. 5.

$u_1 - u_0 = \frac{1}{2}(2 + e^{2u_0} + e^{-2u_1}) > 0$ from (9) ; therefore dx_1/dt is positive, and t increases with x_1. Again,

$$\frac{dy_1}{dt} = y_0 \frac{\operatorname{ch} u_0 \operatorname{sh} u_1 \dfrac{du_1}{dt} - \operatorname{ch} u_1 \operatorname{sh} u_0 \dfrac{du_0}{dt}}{\operatorname{ch}^2 u_0} \tag{18}$$

$$= y_0 \frac{\operatorname{ch}^2 u_0 \operatorname{sh} u_1(\operatorname{ch} u_1 + \operatorname{sh} u_1) - \operatorname{ch}^2 u_1 \operatorname{sh} u_0(\operatorname{ch} u_0 - \operatorname{sh} u_0)}{4 \operatorname{ch} u_1 \operatorname{ch}^3 u_0} \tag{19}$$

$$= y_0 \frac{\operatorname{ch}^2 u_0 \operatorname{sh}^2 u_1 + \operatorname{ch}^2 u_1 \operatorname{sh}^2 u_0 + \operatorname{ch} u_1 \operatorname{ch} u_0 \operatorname{sh}(u_1 - u_0)}{4 \operatorname{ch} u_1 \operatorname{ch}^3 u_0}. \tag{20}$$

Therefore $\frac{dy_1}{dt} > 0$, and t increases with y_1. Then from (17) and (20),

$$\frac{dy_1}{dx_1} = \operatorname{ch} u_1 + \operatorname{sh} u_1 = e^{u_1} > 0, \tag{21}$$

and y_1 increases with x_1. Further,

$$\frac{d_2y_1}{dx_1^2} = e^{u_1}\frac{du_1}{dx_1} = e^{u_1}\cdot\frac{du_1}{dt}\cdot\frac{dt}{dx_1}, \tag{22}$$

where $e^{u_1} > 0$, $\frac{du_1}{dt} > 0$ from (13), and $\frac{dt}{dx_1} > 0$ from (17).

Therefore $\frac{d_2y_1}{dx_1^2} > 0$ and the curve represented by equations (7), (8), (9) is convex to the x-axis.

As t increases from $-\infty$ to $+\infty$, x_1 and y_1 both increase continually; as t approaches $-\infty$, i. e. as u_0 and u_1 (see table I) approach $-\infty$, x_1 and y_1 approach zero. This is proved as follows:

$$x_1 = y_0\frac{u_1 - u_0}{\operatorname{ch} u_0} = y_0\frac{2 + e^{2u_0} + e^{-2u_1}}{e^{u_0} + e^{-u_0}}$$

$$= y_0\frac{e^{-(2u_1 - u_0)}\,(2e^{2u_1} + e^{2u_0 + 2u_1} + 1)}{e^{2u_0} + 1}. \tag{23}$$

Now $u_0 - u_1 = -\frac{1}{2}(2 + e^{2u_0} + e^{-2u_1})$, from (9); therefore

$$u_0 - 2u_1 = -\tfrac{1}{2}(2 + e^{2u_0} + 2u_1 + e^{-2u_1}).$$

Again,

$$2u_1 + e^{-2u_1} = \frac{1 + 2u_1\,e^{2u_1}}{e^{2u_1}},$$

which approaches $+\infty$ as u_1 approaches $-\infty$, since $2u_1e^{2u_1}$ approaches 0. Therefore $u_0 - 2u_1$ approaches $-\infty$ as u_1 and u_0 approach $-\infty$. Then from equation (23), x_1 approaches 0 as u_1 and u_0 approach $-\infty$.

Moreover, $y_1 = y_0\frac{\operatorname{ch} u_1}{\operatorname{ch} u_0} = y_0\frac{e^{u_1} + e^{-u_1}}{e^{u_0} + e^{-u_0}}$

$$= y_0\frac{e^{-(u_1 - u_0)}\,(e^{2u_1} + 1)}{e^{2u_0} + 1}.$$

Therefore y_1 approaches 0 as u_1 and u_0 approach $-\infty$, since

$$u_1 - u_0 = \tfrac{1}{2}\,(2 + e^{2u_0} + e^{-2u_1})$$

approaches $+\infty$. Therefore the graph of our curve starts at the origin.

Similarly, as t approaches $+\infty$, i. e. as u_1 and u_0 approach $+\infty$ (see table I), x_1 and y_1 approach $+\infty$; for,

$$x_1 = y_0\,\frac{u_1 - u_0}{\operatorname{ch} u_0} = y_0\,\frac{(2 + e^{2u_0} + e^{-2u_1})}{e^{u_0} + e^{-u_0}} = y_0\,\frac{2e^{-u_0} + e^{u_0} + e^{-2u_1 - u_0}}{1 + e^{-2u_0}},$$

so that x_1 approaches $+\infty$ as u_1 and u_0 approach $+\infty$;

and
$$y_1 = y_0\,\frac{e^{u_1} + e^{-u_1}}{e^{u_0} + e^{-u_0}} = y_0\,\frac{e^{u_1 - u_0} + e^{-u_1 - u_0}}{1 + e^{-2u_0}},$$

which approaches $+\infty$ as u_1 and u_0 approach $+\infty$, since $u_1 - u_0$ approaches $+\infty$.

Taking corresponding values of u_0 and u_1 from the graph given in Figure 5 we obtain the following table of corresponding values of x_1 and y_1, for $x_0 = 0$ and $y_0 = 1$.

TABLE II

u_1	t	u_0	x_1	y_1
$-\infty$	$-\infty$	$-\infty$	.00	.00
–1.00	–10.39	–5.14	.05	.02
.00	–2.00	–1.55	.63	.41
.50	–0.36	–0.80	.97	.84
1.00	+0.86	–0.33	1.26	1.46
1.50	1.88	–0.04	1.54	2.35
2.00	2.90	+0.21	1.75	3.68
2.50	3.91	0.39	1.96	5.69
3.00	4.93	0.55	2.12	8.66
.	.	.	.	.
.	.	.	.	.
∞	∞	∞	∞	∞

With the aid of this table we construct the graph $\mathfrak{C}$ determined by equa-

tions (7), (8), (9), and on the same figure we draw the curve $\mathfrak{F}$ given in Figure 2 of the preceding paper.

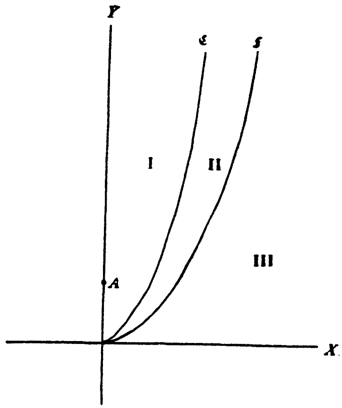

Fig. 6.

From our results in the two papers we know that from the point A to points of regions I and II two catenaries can be drawn with the x-axis as directrix; of these the upper one furnishes a relative minimum in the problem of the minimum surface of revolution;* to points of curve $\mathfrak{F}$ one catenary can be drawn, and to points of region III no catenaries can be drawn.

Then considering the two solutions of the problem of the minimum surface of revolution; i. e. the catenary solution and the discontinuous solution, we can conclude as follows:

1) *In the region* I *both the catenary solution and the discontinuous solution give a relative minimum, and the surface of the catenary is the smaller.*
2) *For points along the curve* $\mathfrak{C}$ *both solutions give a relative minimum and the surfaces are equal.*
3) *In the region* II *both solutions give a relative minimum and the discontinuous solution is the smaller.*
4) *Along the curve* $\mathfrak{F}$ *the catenary solution does not furnish a minimum,*† *so that the discontinuous solution is the only solution.*
5) *In the region* III *the discontinuous solution is the only solution.*

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* Todhunter, *Researches in the Calculus of Variations*, p. 57; Hancock, *Calculus of Variations*, chapters II and III.

† Lindelöf's Theorem; see Hancock, "On the number of catenaries which may be drawn through two fixed points," Annals of Mathematics, ser. 1, vol. 10, p. 159, or *Calculus of Variations*, chapter III.